International Research Training Group IRTG 2657
Computational Mechanics Techniques in High Dimensions

Model Reduction for Fluid-Structure Interactions involving Moving Surfaces

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Motivation

The Fluid-Structure Interaction, an essential but complex multiphysics coupling in engineering, involves major challenges:

- *strong coupling* (immediate mutual interaction between fluid and structure);
- *different formulations:* solids are described by following the motion of matter (Lagrangian formulation), while fluids are studied from a fixed point in space (Eulerian formulation);
- *large deformations* and *complex boundary conditions* that *evolve over time* (e.g.: walls of a pipe that move or change shape);
- *non-linear problems,* requiring a fine mesh that accurately represents certain phenomena (e.g. vortexes formed in a fluid flow).

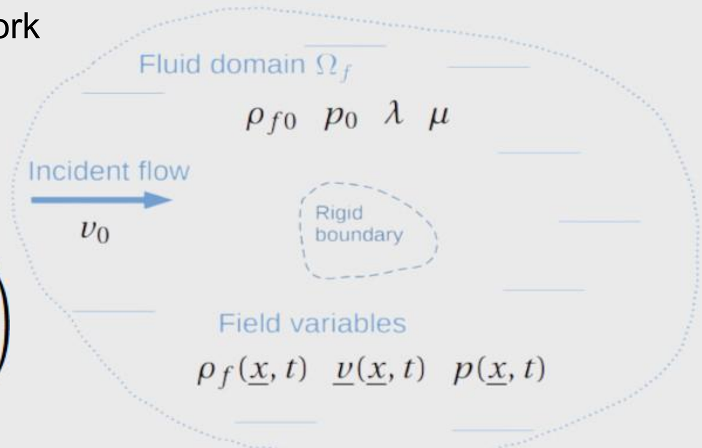
To date, modeling a complex system in FSI (airbag deployment, wind turbine dynamics, underwater pipeline behavior, etc.) requires massive computing resources, while producing results whose quality often leaves room for improvement. The objective of this thesis is therefore to propose an innovative approach that is both accurate and economical in terms of computing resources.

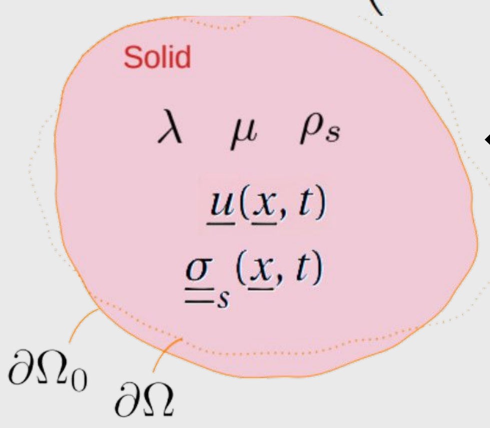
Governing Equations

- **Description of the fluid flow** in a fixed-grid Eulerian framework

$$\begin{cases} \rho_f \dfrac{\partial \mathbf{v}}{\partial t} + \rho_f\ \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{f}_{\text{exc}}(\mathbf{x},t) \quad \text{in } \Omega_f \\ \nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega_f \end{cases}$$

with $\mathbf{f}_{exc}(x,y,t) = \begin{pmatrix} \exp\left(-\frac{(x-x_c)^2}{2\sigma_x^2} - \frac{(y-y_c)^2}{2\sigma_y^2}\right)\cdot\exp\left(-\frac{(t-t_0)^2}{2\sigma_t^2}\right) \\ 0 \end{pmatrix}$





- **Description of the structural motion** (E-B beam) in a Lagrangian framework

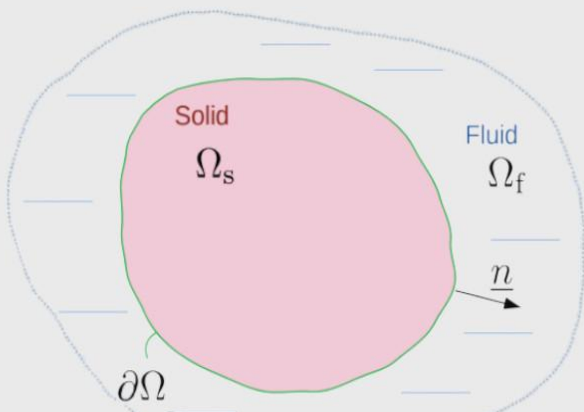
$$\rho_s A\frac{\partial^2 u}{\partial t^2} + EI\frac{\partial^4 u}{\partial y^4} = f_{\text{fs}}(x,t) \quad \text{in } \Omega_s \quad \text{with } f_{fs} = (\boldsymbol{\sigma}_s\cdot\mathbf{n})\cdot\boldsymbol{e}_x$$

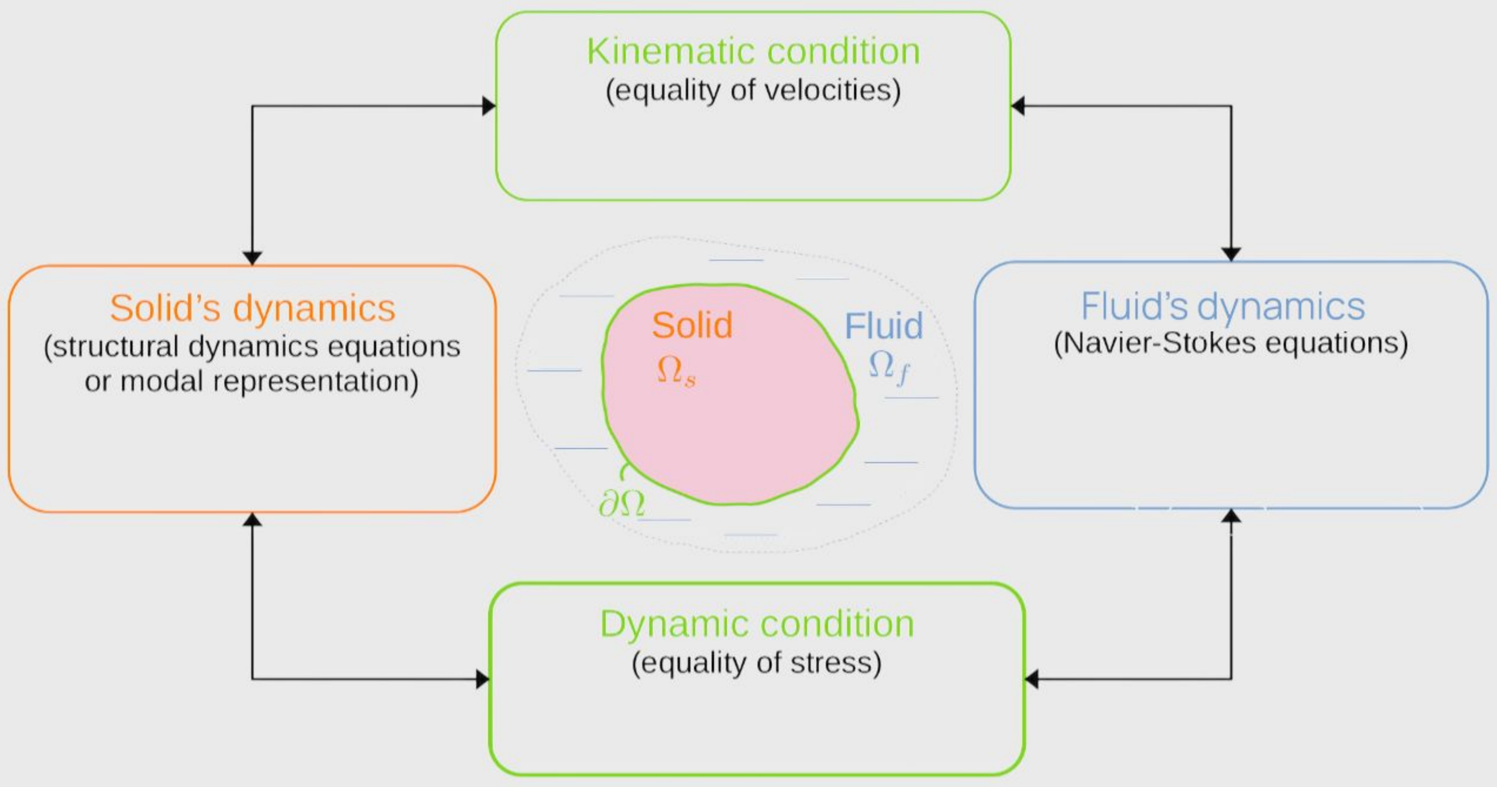
- **FSI-coupling conditions**
  - *kinematic condition : equality of velocities* on the fluid/solid interface

$$\mathbf{v}\cdot\mathbf{n} = \frac{d\mathbf{u}}{dt}\cdot\mathbf{n} \quad \text{in } \partial\Omega$$

  - *dynamic condition : equality of stress* on the fluid/solid interface

$$\boldsymbol{\sigma}_s\cdot\mathbf{n} = \boldsymbol{\sigma}_f\cdot\mathbf{n} \quad \text{in } \partial\Omega \quad \text{with } \boldsymbol{\sigma}_f = -p\mathbf{I} + \mu\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right)$$





Finite Element Formulation

- **Fluid** $\quad \boldsymbol{M}_f\dot{\boldsymbol{V}}_f^{n+1} + \boldsymbol{K}_f\boldsymbol{V}_f^{n+1} - \boldsymbol{B}_f^T\boldsymbol{P}_f^{n+1} = \boldsymbol{F}_{exc}^{n+1}$ and $\boldsymbol{B}_f\boldsymbol{V}_f^{n+1} = 0$
- **Solid**

$$\underbrace{\left[\frac{1}{\Delta t^2}\mathbf{M}_s + \frac{1}{\Delta t}\mathbf{C}_s + \mathbf{K}_s\right]}_{\mathbf{A}_s}\mathbf{U}_s^{n+1} = \frac{1}{\Delta t^2}\mathbf{M}_s\left(2\mathbf{U}_s^n - \mathbf{U}_s^{n-1}\right) + \frac{1}{\Delta t}\mathbf{C}_s\mathbf{U}_s^n + \mathbf{F}_{fs}$$

- **Coupling (in ∂Ω)**

$$\begin{cases} \mathbf{L}_f\cdot\mathbf{V}_f^{n+1} = \mathbf{L}_s\dfrac{\mathbf{U}_s^{n+1}-\mathbf{U}_s^n}{\Delta t} \\ \mathbf{F}_{fs} = \underbrace{\mu\mathbf{L}_f\cdot\mathbf{V}_f^{n+1}}_{\text{viscous force}} + \underbrace{\mathbf{L}_s\cdot\mathbf{P}_f^{n+1}}_{\text{pressure force}} \end{cases}$$

Model Order Reduction (PGD)

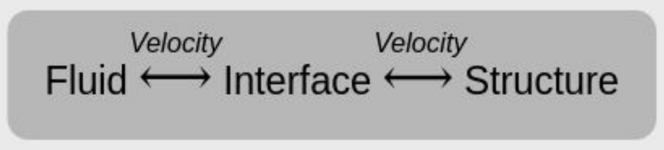
Time-space decomposition [2] for $v_f(\mathbf{x},t)$, $p_f(\mathbf{x},t)$ and $\mathbf{u}_s(\mathbf{x},t)$:

$$\forall(\mathbf{x},t)\in\Omega\times[0,T],\quad u(\mathbf{x},t) \approx \sum_{i=1}^{N_s}\phi_i(\mathbf{x})\,\lambda_i(t)$$

- ➔ Spatial basis $\{\phi_i(\mathbf{x})\}_{i=1\ldots m}$ built on-the-fly with a greedy algorithm
- ➔ No need for prior information (snapshots)

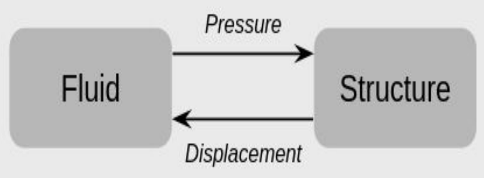
Different Coupling Approaches

Monolithic (strong): solve the fluid and solid problems simultaneously [1]



$$\begin{bmatrix} \mathbf{A}_f & \mathbf{B}_f^T & \mathbf{0} & \mathbf{C}_{kin}^T \\ \mathbf{B}_f & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_s & -\frac{1}{\Delta t}\mathbf{I} \\ \mathbf{C}_{kin} & \mathbf{0} & -\frac{1}{\Delta t}\mathbf{I} & \mathbf{0} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{V}_f^{n+1} \\ \mathbf{P}_f^{n+1} \\ \mathbf{U}_s^{n+1} \\ \boldsymbol{\lambda}^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t}\mathbf{M}_f\cdot\mathbf{V}_f^n + \mathbf{F}_t \\ \mathbf{0} \\ \frac{1}{\Delta t^2}\mathbf{M}_s\cdot(2\mathbf{U}_s^n - \mathbf{U}_s^{n-1}) + \frac{1}{\Delta t}\mathbf{C}_s\cdot\mathbf{U}_s^n \\ -\frac{1}{\Delta t}\mathbf{U}_s^n \end{bmatrix}$$

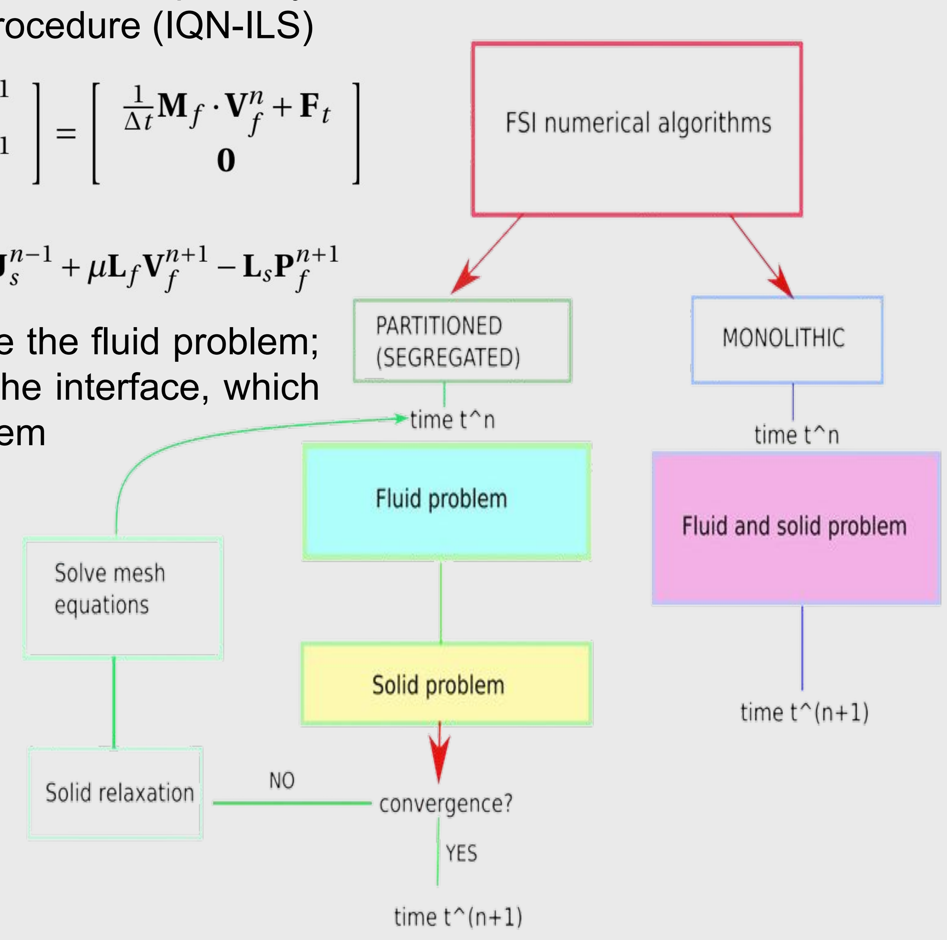
Partitioned (strong): solve the fluid and solid problems separately, then couple the two physics using an iterative procedure (IQN-ILS)



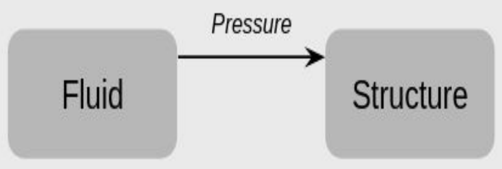
$$\begin{bmatrix} \mathbf{A}_f & \mathbf{B}_f^T \\ \mathbf{B}_f & \mathbf{0} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{V}_f^{n+1} \\ \mathbf{P}_f^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t}\mathbf{M}_f\cdot\mathbf{V}_f^n + \mathbf{F}_t \\ \mathbf{0} \end{bmatrix}$$

$$\updownarrow$$

$$\mathbf{A}_s\mathbf{U}_s^{n+1} = \mathbf{b}_s\mathbf{U}_s^n + \mathbf{c}_s\mathbf{U}_s^{n-1} + \mu\mathbf{L}_f\mathbf{V}_f^{n+1} - \mathbf{L}_s\mathbf{P}_f^{n+1}$$



Unidirectional (weak): at each time t, first solve the fluid problem; then extract the fluid velocity and pressure on the interface, which will act as the second member in the solid problem



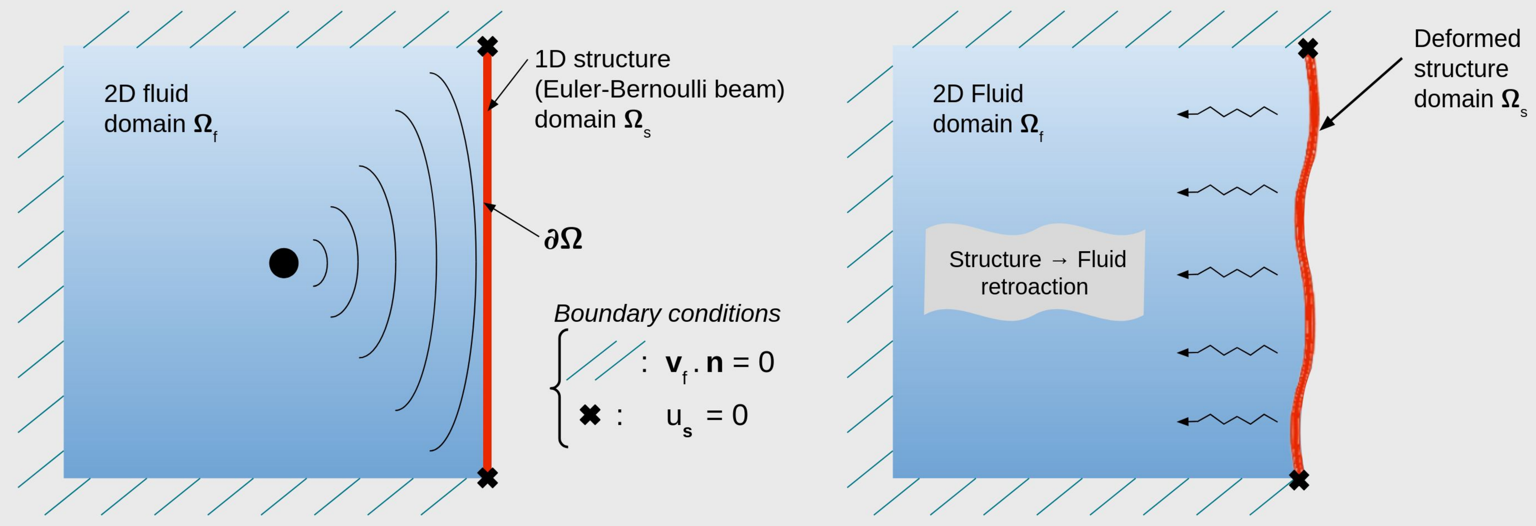
$$\begin{bmatrix} \mathbf{A}_f & \mathbf{B}_f^T \\ \mathbf{B}_f & \mathbf{0} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{V}_f^{n+1} \\ \mathbf{P}_f^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t}\mathbf{M}_f\cdot\mathbf{V}_f^n + \mathbf{F}_t \\ \mathbf{0} \end{bmatrix}$$

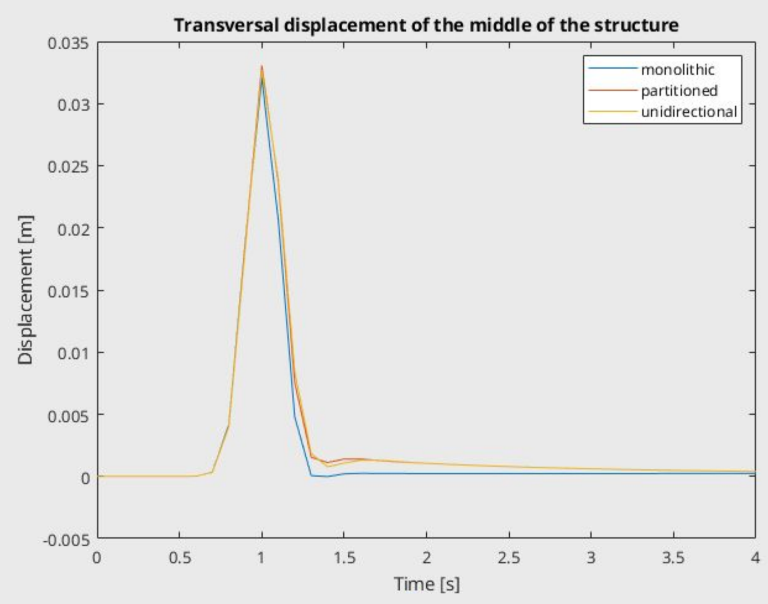
$$\downarrow$$

$$\mathbf{A}_s\mathbf{U}_s^{n+1} = \mathbf{b}_s\mathbf{U}_s^n + \mathbf{c}_s\mathbf{U}_s^{n-1} + \mu\mathbf{L}_f\mathbf{V}_f^{n+1} - \mathbf{L}_s\mathbf{P}_f^{n+1}$$

First Results

- **Model:** a brief Gaussian excitation at the center of a square fluid domain (2D) deforms the beam structure placed at the right edge of the domain. The convective term of the Navier-Stokes equations is neglected.





Computational time of the different approaches:

Coupling approach	Computational time (s)
Monolithic	30.545
Partitioned (IQN-ILS)	29.676
Unidirectional (fluid → solid)	28.046

Next Steps

- Compare the different coupling approaches' results to a numerical reference solution (COMSOL)
- Add PGD to these monolithic, partitioned, and unidirectional numerical models [2];
- Design reduction algorithms adapted to the specific characteristics of fluid and solid media, ensuring both numerical efficiency and physical fidelity, taking into account the effects related to the nature of the fluid (light or heavy) and the structure (thin or massive), using a fine and controlled coupling approach.
- Implement the extended PGD methodology [3], applied to an evolving geometry, and carry out initial comparisons.

References

[1] Boffi D., Cavallini N. and Gastaldi L., The Finite Element Immersed Boundary Method With Distributed Lagrange Multiplier, SIAM Journal on Numerical Analysis, Vol. 53, No. 6 (2015), pp. 2584-2604
[2] Ladevèze P., Nonlinear computational structural mechanics new approaches and non-incremental methods of calculation. Springer, 1999
[3] Beckermann M., Cremonesi M. and Barbarulo A., Data reduction strategies for the Particle Finite Element Method, a Lagrangian fluid solver, Elsevier, Mar. 2025

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